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ZF AND BOOLEAN ALGEBRA[†, ††]

BY

J. M. PLOTKIN

ABSTRACT

A generically embedded Boolean algebra is studied. Several results about infinite complete Boolean algebras are shown to be independent of ZF.

0.

The study of Boolean algebras would lose much of its glamour if it were denied the topological duality theory introduced by M. H. Stone [21]. Fundamental to this theory is the Boolean Prime Ideal theorem which is a consequence of the axiom of choice. In [7] Feferman showed the Prime Ideal theorem to be independent of Zermelo-Fraenkel set theory (ZF).

The axiom of choice makes its presence felt throughout the theory of Boolean algebras. For example in ZF with the axiom of choice (ZFC) every infinite Boolean algebra has a countably infinite disjointed subset [5, p. 7], and hence every infinite complete Boolean algebra has a subalgebra isomorphic to $P(\omega)$, the power set of ω [20, p. 66]. Numerous other such consequences of ZFC can be gleaned from [5, 11, 20]. Assuming the consistency of ZF, one can ask whether these results are independent of ZF. In this paper we show that many are.

In Section 1 we construct a "choiceless" model of ZF via generic embedding of an atomless Boolean algebra and we study some of the basic properties of this model. In Section 2 we prove that the embedded algebra is complete and we discuss its free subalgebras. In Section 3 we study two types of decompositions of complete Boolean algebras: direct unions and inverse limits.

Our basic references are: [20] for Boolean algebra and [3] for logic. All terms which we leave undefined can be found in these works.

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1.

We refer the reader to [17] for a description of generic embedding. We now briefly summarize the major result of [17].

T denotes a complete first order theory with equality which has infinite models and is $\aleph_0$-categorical. $L(T)$ and $F(T)$ denote the language of T and the formulas in this language, respectively. For B a model of T, $D_n(B)$ denotes the set of all n-ary relations on B which are definable by elements of $F(T)$, possibly with parameters from B, i.e., if

$$S = \{\langle b_1, \cdots, b_n \rangle \mid \quad B \models \varphi(b_1, \cdots, b_n, a_1, \cdots, a_m)\}$$

where $\varphi(y_1, \cdots, y_n, x_1, \cdots, x_m) \in F(T)$ and $a_i \in B$ $1 \leqq i \leqq m$, then $S \in D_n(B)$.

CONVENTION. We write $D(B)$ instead of $D_1(B)$.

M is a countable standard transitive model of ZFC. In M there is a countable (in M) model $\tilde{A}$ of T.

DEFINITION.

i) A set S is finite if there is a bijection between S and some ordinal less than ω. Otherwise S is infinite.

ii) A set S is Dedekind finite if it has no infinite well orderable subsets, or equivalently there are no bijections between S and any of its proper subsets.

iii) For any set S, $|S|$ denotes the cardinality of S. $|S|$ is said to be finite, infinite, or Dedekind finite just in case S is.

We now give the main theorem of [17].

THEOREM 1.0. *There is a model N of* ZF *such that N is a generic extension of M and such that N contains an isomorphic (in the real world) copy A of $\tilde{A}$. A is infinite, Dedekind finite, and the set of n-ary relations on A in N is just $D_n(A)$.*

For the remainder of this paper T will denote the first order theory of atomless Boolean algebras. $L(T)$ has as its nonlogical symbols binary operation symbols $\cap$, $\cup$, a unary operation symbol $-$, and constants $0, 1$. We introduce the following abbreviations for certain elements of $F(T)$: we write $t_1 \neq t_2$ for $\neg(t_1 = t_2)$, $t_1 \leqq t_2$ for $t_1 \cap t_2 = t_1$, and $t_1 < t_2$ for $t_1 \cap t_2 = t_1 \wedge \neg(t_1 = t_2)$. In what follows we let the context differentiate the formal symbols from their intended interpretations. B will always denote a Boolean algebra. For $S \subseteq B$ we let $\langle S \rangle$ denote the subalgebra generated by S. For $a \in B$ a^0 is a and a^1 is $\bar{a}$ (we also apply this convention to terms of $L(T)$).

T is complete and $\aleph_0$-categorical [3] so the methods of [17] apply. Let M be a countable standard transitive model of ZFC — we assume such things exist — and let $\tilde{A}$ be (in M) a countable atomless Boolean algebra. Let N and $A \in N$ be as guaranteed by Theorem 1.0. In N, A is an infinite, Dedekind finite atomless Boolean algebra. (Thus being atomless is compatible with at least one kind of "finiteness".).

We devote ourselves to ferreting out the properties that A enjoys relative to N. These properties will be stated as propositions concerning A. In general these statements will be followed by remarks giving the known situation in ZFC. The overall result will be that certain facts about Boolean algebras are independent of ZF.

Our first result is an observation based on work of J. Grant [8] and K. McAloon [14].

PROPOSITION 1.1. *A has no non-trivial automorphism.*

PROOF. Any automorphism of A belongs to $D_2(A)$ by Theorem 1.0. Grant has shown that any two models of a complete theory have the same group of definable automorphisms. In [14] McAloon remarks that there exist atomless Boolean algebras which have no non-trivial automorphisms. In particular these algebras have no non-trivial definable automorphisms. By absoluteness of the necessary notions A is a "real world" model of T. Hence the $D_2(A)$ automorphisms of A are trivial.

Since A is both infinite and Dedekind finite, the axiom of choice is false in N. Actually the nature of T and A enable us to show that the axiom of choice for sets of unordered pairs is false in N.

We need some preliminary definitions and lemmas.

DEFINITION. $S \subseteq B$. S is *independent* if for each finite subset $\{s_1, \cdots, s_n\} \subseteq S$ and each sequence $(\gamma_1, \cdots, \gamma_n) \in {}^n\{0,1\}$, $s_1^{\gamma_1} \cap \cdots \cap s_n^{\gamma_n} \neq 0$. Further for $b \in B$ and $S \subseteq B$, b is *independent of* S if for each $s \in S$, $s \neq 0$, $s \neq 1$ we have $\{b, s\}$ is independent.

LEMMA 1.2. *$b \in B$, S a subalgebra of B, b independent of S. Let $B' = \langle S \cup \{b\} \rangle$. There is an automorphism α of B' such that $\alpha(s) = s$ for all $s \in S$ and $\alpha(b) = \bar{b}$.*

PROOF. This lemma is a version of lemma 22.15 of [5]. We sketch its proof. Since b is independent of S, every element of B' can be uniquely written as $(s_1 \cap b) \cup (s_2 \cap \bar{b})$ where $s_1, s_2 \in S$. Define α by

$$\alpha((s_1 \cap b) \cup (s_2 \cap \bar{b})) = (s_2 \cap b) \cup (s_1 \cap \bar{b}).$$

α is the desired automorphism.

LEMMA 1.3. *B atomless. $S \subseteq B$ a finite subalgebra. There is a $b \in B$ such that b is independent of S.*

PROOF. Let $s_1, \cdots, s_n$ be the atoms (in sense of S) which generate S. Since b is atomless for each i, $1 \leqq i \leqq n$, there is a $b_i \in B$ with $0 < b_i < s_i$. Let $b = b_1 \cup \cdots \cup b_n$. It is easily verified that b is independent of S.

LEMMA 1.4. *B countable and atomless. $S \in D(B)$. There is a $b \in B$ such that either $b, \bar{b} \in S$ or $b, \bar{b} \notin S$.*

PROOF. Assume $S = \{b \mid B \models \Psi(b, a_1, \cdots, a_m)\}$ where $\Psi(y, x_1, \cdots, x_m) \in F(T)$ and $a_i \in B$ $1 \leqq i \leqq m$. Let $S' = \langle\{a_1, \cdots, a_m\}\rangle$. By Lemma 1.3 there is a $b \in B$ which is independent of S' (hence $\bar{b}$ is also independent of S'). We now prove that if $b \in S$ then $\bar{b} \in S$. If $b \in S$ then $B \models \Psi(b_1 a_1, \cdots, a_m)$. From Lemma 1.2 there is an automorphism α of $\langle S' \cup \{b\}\rangle$ such that $\alpha(a_i) = a_i$, $1 \leqq i \leqq m$, and $\alpha(b) = \bar{b}$. Since B is countable and atomless, a zig-zag argument shows that α can be extended to an automorphism β of B (see [1] for such an argument for atomless Boolean rings). Alternatively we can make the following observations: T is $\aleph_0$-categorical and since B is a countable model of T, B is an atomic model [3, p. 93]. Hence $(a_1, \cdots, a_m, b)$ satisfies some atom of $F_{m+1}(T)$, the formulas of T with at most $m + 1$ free variables. But T admits an elimination of quantifiers (see Section 2, Lemma 2.0). Thus $(a_1, \cdots, a_m, b)$ satisfies a quantifier free atom of $F_{m+1}(T)$. The existence of α shows that $(a_1, \cdots, a_m, \bar{b})$ satisfies the same atom. Since B is atomic and countable there is an automorphism β of B such that $\beta(a_i) = a_i$, $i \leqq i \leqq m$, and $\beta(b) = \bar{b}$. In any case we have $B \models \Psi(\beta(b), \beta(a_1), \cdots, \beta(a_m))$. Thus $B \models \Psi(\bar{b}, a_1, \cdots, a_m)$ and $\bar{b} \in S$. A similar argument works if $\bar{b} \in S$. Thus either $b, \bar{b} \in S$ or $b, \bar{b} \notin S$.

PROPOSITION 1.5. *The axiom of choice for sets of unordered pairs is false in N.*

PROOF. Consider A. Let $C = \{\{a, \bar{a}\} \mid a \in A\}$. A choice set S for C would be a subset of A such that for each $a \in A$ either $a \in S$ or $\bar{a} \in S$ but not both. By Theorem 1.0 $S \in D(A)$. For some $\Psi \in F(T)$ we would have

$$A \models \text{"For all } x \text{ either } \Psi(x) \text{ or } \Psi(\bar{x}) \text{ but not both"}.$$

Now the same sentence must also be true in $\tilde{A}$. This is impossible by Lemma 1.4. Thus no such S exists.

COROLLARY 1.6. *A has no ultrafilters.*

PROOF. Amongst other things an ultrafilter serves as a choice set for the above set C.

REMARKS. For B, $S(B)$ denotes the set of ultrafilters of B. If B is finite, $|S(B)|=n$ where n is the number of atoms of B. For B infinite, Makinson [13] and Grätzer [9] have shown in ZFC that $|S(B)|\geqq|B|$. Corollary 1.6 shows this result cannot be proven in ZF. This conclusion could also be drawn from [7] where Feferman constructs a model of ZF in which $P(\omega)$ has non nonprincipal ultrafilters. If F is the filter of cofinite subsets of ω, then the Boolean algebra $P(\omega)/F$ in Feferman's model is infinite and has no ultrafilters. This algebra, however, is far from Dedekind finite since $P(\omega)$ contains a family of infinite subsets of ω of cardinality of the continuum which are pairwise almost disjoint [19, p. 81].

2.

In order to gain more insight into the properties of A we need to study its subsets. Fortunately this reduces to an examination of $D(A)$. Our task is made less burdensome by the fact that T admits an elimination of quantifiers.

LEMMA 2.0. *Let* $\Psi(x_1,\cdots,x_n)\in F(T)$ *with free variables* $x_1,\cdots,x_n$. *Then* $T\vdash\Psi(x_1,\cdots,x_n)\leftrightarrow\bigvee_j(\bigwedge_i\alpha_{ij}(x_1,\cdots,x_n))$ *where each* α_{ij} *is either* i) $x_1^{\gamma_1}\cap\cdots\cap x_n^{\gamma_n}=0$ *or* ii) $x_1^{\gamma_1}\cap\cdots\cap x_n^{\gamma_n}\neq 0$ *where* $\gamma_i\in\{0,1\}$.

PROOF. (Indication.) Proceed by induction on logical complexity of formulas. Use some elementary facts on Boolean algebras to insure the presence of each free variable in each conjunct and use the usual quantifier elimination techniques as described in [3].

LEMMA 2.1. B *atomless,* $b_i\in B$, $1\leqq i\leqq n$. $C=\langle\{b_1,\cdots,b_n\}\rangle$, $\Psi(y,x_1,\cdots,x_n)\in F(T)$. *Let* T_1 *be the inessential expansion of* T *obtained by adding constants* b_j, c_j, d_j, c_{ij} *for the elements of* C. *Then*

$$T_1\vdash\Psi(y,b_1,\cdots,b_n)\leftrightarrow\bigvee_{1\leqq j\leqq n}(y\cap c_j=0\wedge\bar{y}\cap d_j=0\wedge\bigwedge_{1\leqq i\leqq t}(y^{\gamma_{ij}}\cap c_{ij}\neq 0)).$$

PROOF. Apply Lemma 2.0 and observe that

$$T_1\vdash y^\gamma\cap a=0\wedge y^\gamma\cap b=0\leftrightarrow y^\gamma\cap(a\cup b)=0.$$

NOTE. In the above disjunction we can always assume that for each j the conjuncts $y \cap c_j = 0$, $\bar{y} \cap d_j = 0$ are present by adding $y \cap 0 = 0$ and $\bar{y} \cap 0 = 0$ if necessary.

LEMMA 2.2. *B atomless, $S \in D(B)$. Then $S = \bigcup_{1 \leqq j \leqq n} S_j$ where either*
i) $S_j = [d_j, a_j] \cap \{b \in B \mid \wedge_i b^{\gamma_{ij}} \cap c_{ij} \neq 0\}$
or
ii) $S_j = [d_j, a_j]$
and where $[d_j, a_j] = \{b \in B \mid d_j \leqq b \leqq a_j\}$.

PROOF. This is just a translation of Lemma 2.1. In this translation a_j is $\bar{c}_j$.

LEMMA 2.3. *B atomless, $a_i \in B$, $a_i \neq 0$, $1 \leqq i \leqq n$. Then there exist f_i with $0 < f_i < a_i$ such that $f_i \cap f_j = 0$ for $i \neq j$.*

PROOF. Induction on n.

LEMMA 2.4. *B atomless, $S \in D(B)$. Then*
i) *the meet and join of S exist;*
ii) *if S is a filter (ideal), then S is principal.*

PROOF. i) First we prove that the meet of S exists. By Lemma 2.2, $S = \bigcup_{1 \leqq j \leqq n} S_j$. Hence to show that the meet of S exists it suffices to show that the meet of S_j exists, $1 \leqq j \leqq n$, for then meet S = meet $S_1 \cap \cdots \cap$ meet S_n. Appealing to Lemma 2.2 we assume that

$$S_j = [d_j, a_j] \cap \left\{ b \in B \mid \bigwedge_i b^{\gamma_{ij}} \cap c_{ij} \neq 0 \right\}$$

and that $I = \{i \mid \gamma_{ij} = 0\} \neq \emptyset$. We leave the other cases to the reader. We claim that the meet of S_j is d_j. If $d_j \in S_j$ we are done. Assume $d_j \notin S_j$. Thus there is a $b \in S_j$ with $d_j < b$. Now $b \cap c_{ij} \neq 0$ for all $i \in I$. By Lemma 2.3 we can choose $\{f_i \mid i \in I\}$ such that $0 < f_i < b \cap c_{ij}$ with $f_{i_1} \cap f_{i_2} = 0$ for $i_1 \neq i_2$. Now using this same lemma choose e_i such that $0 < e_i < f_i$. Let $e = d_j \cup (\bigcup_{i \in I} e_i)$ and let $f = d_j \cup (\bigcup_{i \in I} (\bar{e}_i \cap f_i))$. If $i \in I$ then $e \cap c_{ij} \neq 0$ and $f \cap c_{ij} \neq 0$. If $i \notin I$, then since $e, f \leqq b$ and $\bar{b} \cap c_{ij} \neq 0$ we have $\bar{e} \cap c_{ij} \neq 0$ and $\bar{f} \cap c_{ij} \neq 0$. Thus, $e, f \in S_j$. By construction $e \cap f = d_j$. Thus since $d_j \leqq b$ for each $b \in S_j$ we have d_j is the meet of S_j. Hence $d_1 \cap \cdots \cap d_n$ is the meet of S. Let $\tilde{S} = \{b \in B \mid \bar{b} \in S\}$. Since $S \in D(B)$, $\tilde{S} \in D(B)$. By the above argument the meet of $\tilde{S}$ exists. But the meet of $\tilde{S}$ equals the join of S. Thus the join of S exists.

ii) If S is a filter the argument of part (i) shows that $d_j \in S$ and hence that $S = \{b \in B \mid d_1 \cap \cdots \cap d_n \leqq b\}$. Note that S is an ideal implies $\tilde{S}$ is a filter. So ideal or filter S must be principal. We now immediately obtain:

PROPOSITION 2.5.

i) *A is a complete Boolean algebra and every subalgebra of A is a complete subalgebra of A.*

ii) *Every filter (ideal) of A is principal.*

PROOF. i) the fact that subalgebras are complete subalgebras (which is more than just complete as a Boolean algebra) follows from the proof of Lemma 2.4 (i).

REMARKS. Pierce has shown in ZFC that an infinite cardinal κ is the cardinality of an infinite complete B if and only if $\kappa^{\aleph_0} = \kappa$ [15]. Since A is Dedekind finite, $|A|^{\aleph_0} \neq |A|$. Also in ZFC infinite complete B have subalgebras isomorphic to $P(\omega)$ [20, p. 66]. The Dedekind finiteness of A implies that A has no such subalgebra. In [6] Dwinger studied ideals of infinite complete B. For principal ideals I, B/I is complete and the natural homorphism $\nu : B \to B/I$ is complete. Dwinger showed that in ZFC an infinite complete B has ideals I, J such that B/I is complete but $\nu : B \to B/I$ is not complete and such that B/J is not complete. Proposition 2.5 (ii) shows that A has no such ideals. Thus all the above ZFC properties of infinite complete Boolean algebras are independent of ZF.

DEFINITION. $S \subseteq B$. S is *disjointed* if for all $s_1, s_2 \in S$ if $s_1 \neq s_2$ then $s_1 \cap s_2 = 0$. B is said to satisfy the countable chain condition (c.c.c.) if $S \subseteq B$ and S disjointed implies $|S| \leqq \aleph_0$.

LEMMA 2.6. *B atomless. $S \in D(B)$ and S infinite then*

i) *S is not disjointed;*

ii) *is not independent.*

PROOF. The proof is an easy variant of that of Lemma 2.4. $S = \bigcup_{1 \leqq j \leqq n} S_j$. Some S_k must be infinite and hence there is a $b \in S_k$ with $d_k < b$. It is easy to construct $e, f \in S_k$ such that $e \cap f \neq 0$ and $e \neq f$. Thus S is not disjointed. Similarly one can construct $e, f \in S_k$ such that $e \cap \bar{f} = 0$ and $e \neq f$. Thus S is not independent.

PROPOSITION 2.7.

i) *A has no infinite disjointed subsets. A has c.c.c.;*

ii) *A has no infinite independent subsets.*

PROOF. Lemma 2.6.

REMARKS. A of course has arbitrarily large finite disjointed subsets and finite independent subsets. In ZFC every infinite B has a countable disjointed subset [5, p. 7].

Independence is intimately related to freeness. If B is free on a set of generators E, then E is an independent set [20, p. 43].

PROPOSITION 2.8. *A has no infinite free subalgebras.*

REMARKS. In ZFC every atomless (or infinite complete) B has an infinite free subalgebra [5, p. 56]. In [18, p. 241] it is reported that Solovay has proven that if B is infinite, complete and has c.c.c. then B has a free subalgebra on $|B|$ generators. More generally, S. V. Kislyakov has shown that for B infinite and complete $|B|$ is the exact upper bound of the cardinalities of the free subalgebras of B [12]. A shows that both these results cannot be proven in ZF.

Free subalgebras are also connected to the notion of a superatomic Boolean algebra. B is said to be superatomic if every subalgebra of B is atomic. Day has shown that in ZFC, B is superatomic if and only if B has no infinite free subalgebras [4]. A shows that this characterization breaks down in ZF.

3.

Our final results deal with decompositions of complete Boolean algebras.

DEFINITION. $a \in B$. $B \restriction a = \{b \in B \mid b \leqq a\}$.
$B \restriction a$ is a Boolean algebra using complementation relative to a. B is said to be weakly homogeneous if for each $a \neq 0$, $a \in B$ we have $|B| = |B \restriction a|$.

Direct unions of Boolean algebras are just direct products of such algebras. Direct unions of complete Boolean algebras have the following internal characterization [15]:

A complete B is isomorphic to the direct union of complete $B_\alpha, \alpha \in I$ if and only if there is a disjointed set $S = \{s_\alpha \mid \alpha \in I\} \subseteq B$ such that the join of S is 1 and $B \restriction s_\alpha$ is isomorphic to B_α for all $\alpha \in I$.

PROPOSITION 3.0. *A is not the direct union of complete weakly homogeneous Boolean algebras.*

PROOF. We use the internal characterization of direct unions. Any disjointed subset of A is finite by Proposition 2.7 (i). Let S be such a disjointed subset whose join is 1. Since A is infinite, $A \restriction s$ is infinite for some $s \in S$. But $A \restriction s$ is also Dedekind finite. Thus for $b \in A \restriction s$, $b \neq 0$, $b \neq s$, we have $(A \restriction s) \restriction b$, which is $A \restriction b$, is a proper subset of $A \restriction s$; and hence $|A \restriction s| \neq |A \restriction b|$ and $A \restriction s$ is not weakly homogeneous.

REMARKS. In ZFC every complete B is the direct union of complete weakly homogeneous Boolean algebras [20, p. 107].

Representations of infinite Boolean algebras as inverse limits of inverse families of Boolean algebras were introduced by Haimo in [10]. To discuss such representations we need some preliminary definitions.

DEFINITION. A partially ordered set $\langle I, \leqq \rangle$ is said to be *upper directed* if any two elements of I have an upper bound in I.

DEFINITION. Let $\langle I, \leqq \rangle$ be an upper directed partially ordered set. A collection of Boolean algebras $\{B_i \mid i \in I\}$ and homomorphisms $\{\varphi_{ij} \mid i \leqq j, i, j \in I\}$ is called an *inverse family of Boolean algebras* if

i) $\varphi_{ij} : B_j \to B_i$;

ii) φ_{ii} is the identity on B_i;

iii) $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$, $i \leqq j \leqq k$.

DEFINITION. The *inverse limit* B^∞ of such an inverse family is the set of all $p \in \Pi_{i \in I} B_i$ such that for all $i \leqq j$, $\varphi_{ij}(p(j)) = p(i)$. B^∞ is a Boolean subalgebra of $\Pi_{i \in I} B_i$. Associated with B^∞ we have certain canonical homomorphisms π_i, $i \in I$, where $\pi_i : B^\infty \to B_i$ and $\pi_i(p) = p(i)$.

B is said to have an inverse limit representation if there is an inverse family $\{B_i \mid i \in I\}$, $\{\varphi_{ij} \mid i, j \in I\}$ and an isomorphism $\Psi: B \to B^\infty$. In this case the mappings $\Psi_i : B \to B_i$ defined by $\Psi_i = \pi_i \circ \Psi$ are called the decomposition homomorphisms of the representation. This terminology was introduced by Beazer in [2] where he showed that by a simple modification of the original inverse family in a representation one can assume that each φ_{ij} and each Ψ_i is surjective. We assume that such is the case for all representations.

Haimo showed that every infinite Boolean algebra has an inverse limit representation [10]. In some sense Haimo's representations are trivial since some of the decomposition homomorphisms are isomorphisms.

Following Beazer we have:

DEFINITION. B is *inversely reducible* if it has an inverse limit representation in which no decomposition homomorphism is an isomorphism.

The following lemma is inspired by [2].

LEMMA 3.1. *B complete and represented by $\{B_i \mid i \in I\}$, $\{\varphi_{ij} \mid i, j \in I\}$ via $\Psi: B \to B^\infty$. If for each $i \in I$ $\operatorname{Ker}\Psi_i = \{x \mid x \leqq b_i\}$ where $b_i \neq 0$, $b_i \in B$, then $\{b_i \mid i \in I\}$ generates a proper nonprincipal filter ∇ of B.*

PROOF. Note that for $i \leqq j$, $\varphi_{ij} \circ \pi_j = \pi_i$ and thus $\varphi_{ij} \circ \Psi_j = \Psi_i$. For $i \leqq j$ consider b_i, b_j. $\varphi_{ij} \circ \Psi_j(b_j) = 0$. Hence $\Psi_i(b_j) = 0$ and $b_j \leqq b_i$. Let $\{i_1, \cdots, i_n\}$ be a finite subset of I. Since I is upper directed, there is a $k \in I$ with $i_j \leqq k$, $1 \leqq j \leqq n$. By our opening remarks we must have $b_k \leqq b_{i_j}$, $1 \leqq j \leqq n$. Hence $b_{i_1} \cap \cdots \cap b_{i_n} \neq 0$. Thus $\{b_i \mid i \in I\}$ has the finite intersection property and generates a proper filter ∇. We claim that the meet of $\{b_i \mid i \in I\}$, which exists since B is complete, is 0. Suppose $c \leqq b_i$ for all $i \in I$. Then $c \in \operatorname{Ker}\Psi_i$ and $\Psi(c)(i) = 0$ for all i. Thus $\Psi(c) = 0$. But then $c = 0$. Since the meet of $\{b_i \mid i \in I\}$ is 0, ∇ cannot be principal.

PROPOSITION 3.2. *A is not inversely reducible.*

PROOF. Suppose A were inversely reducible via $\Psi: A \to B^\infty$, B^∞ the inverse limit of $\{B_i \mid i \in I\}$, $\{\varphi_{ij} \mid i, j \in I\}$. By inverse reducibility we have that no Ψ_i is an isomorphism. Thus $\operatorname{Ker}\Psi_i \neq \{0\}$ for each i. By Proposition 2.5 (ii) every ideal of A is principal. Hence $\operatorname{Ker}\Psi_i = \{x \mid x \leqq a_i\}$, $a_i \neq 0$. By Lemma 3.1, which holds in N, $\{a_i \mid i \in I\}$ generates a nonprincipal filter of A. But again by Proposition 2.5 (ii) every filter of A is principal. This contradiction shows that A is not inversely reducible.

REMARKS. Beazer proved that every infinite complete Boolean algebra is inversely reducible [2]. His proof used the Prime Ideal theorem. Hence by Proposition 3.2 Beazer's result is independent of ZF.

The referee kindly informs us of overlap of this work with that of U. Felgner. Felgner, in his Habilitations-Schrift of 1971, constructed a model N in which the following held: orderextension, choice for sets of wellorderable sets, countable unions of countable sets are countable, the union of a wellordered family of wellorderable sets is wellorderable, the Kinna-Wagner-Kuratowsky choice axiom, $\neg$BPI, and $\neg AC^\omega$. Further, N contains a Boolean algebra B which is infinite, Dedekind-finite, not isomorphic to a field of sets, has no prime ideal, is rigid, has no infinite independent subset, and is atomless and complete.

REFERENCES

1. A. Abian, *Categoricity of denumerable atomless Boolean rings*, Studia Logica **30** (1972), 63–67.

2. R. Beazer, *An inverse limit representation for complete Boolean algebras*, Glasgow Math. J. **13** (1972), 164–166

3. C. C. Chang, and H. J. Keisler, *Model Theory*, North Holland, 1973.

4. G. W. Day, *Free complete extensions of Boolean algebras*, Pacific J. Math. **15** (1965), 1145–1151.

5. Ph. Dwinger, *Introduction to Boolean Algebras*, Physica-Verlag, Würzburg, 1971.

6. Ph. Dwinger, *On the completeness of the quotient algebras of a complete Boolean algebra II*, Indag. Math., **21** (1959), 26–35.

7. S. Feferman, *Some applications of the notion of forcing and generic* sets, Fund. Math. **56** (1965), 325–345.

8. J. Grant, *Automorphisms definable by formulas*, Pacific J. Math. **44** (1973), 107–115.

9. G. Grätzer, *Lectures on Lattice Theory*, Vol. 1, W. H. Freeman, San Francisco, 1971.

10. F. Haimo, *Some limits of Boolean algebras*, Proc. Amer. Math. Soc. **2** (1951), 566–576.

11. P. R. Halmos, *Lectures on Boolean Algebras*, Van Nostrand, 1967.

12. S. V. Kislyakov, *Free subalgebras of complete Boolean algebras and spaces of continuous functions*, Siberian Math. J. **14**(3) (1973).

13. D. C. Makinson, *On the number of ultrafilters of an infinite Boolean algebra*, Z. Math. Logik Grundlagen Math. **15** (1969), 121–122.

14. K. McAloon, *Consistency results about ordinal definability*, Ann. Math. Logic **2** (1970), 449–467.

15. R. S. Pierce, *A note on complete Boolean algebras*, Proc. Amer. Math. Soc. **9** (1958), 892–896.

16. R. S. Pierce, *Some questions about complete Boolean algebras*, Proc. Symp. Pure Math. **2** (1961), 129–140.

17. J. M. Plotkin, *Generic embeddings*, J. Symbolic Logic **34** (1969), 388–394.

18. H. P. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$*, Acta Math. **124** (1970), 205–248.

19. W. Sierpinski, *Cardinal and Ordinal Numbers*, Warsaw, 1965.

20. R. Sikorski, *Boolean Algebras*, Academic Press, 1964.

21. M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **91** (1937), 375–481.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824 U.S.A.